Optimal Control
Linear Quadratic Regulator

The linear quadratic regulator is a regulator that allows you to specify "energy ratios" between states and the input (which is linearly related to the states), and even between cross correlations between states.

Let $J=\int_0^\infty \boldsymbol{x}^{\boldsymbol{T}}(t)\boldsymbol{Q}\,\boldsymbol{x}(t)+\boldsymbol{u}^{\boldsymbol{T}}(t)\boldsymbol{R}\boldsymbol{u}(t)\,dt$ be the measure you wish to minimize, by picking the state feedback $u(t)=-\boldsymbol{K}\,\boldsymbol{x}(t)$. For this to work ***Q*** and ***R*** must be positive definite.[1] [2] This is required so that the integrand of *J* is always positive. You can use the MATLAB/octave function:
K = lqr(A, B, Q, R)
to get the state feedback that minimizes *J* for you automagically, and how to do that will be discussed after you work through the derivation of the optimum ***K*** for a single state system following the outline below.

1. Write ***J*** for the single state system.
2. Write the state equation for the single state system.
3. It would be nice if you could find a constant matrix ***P***, such that $\frac{-d}{dt}(\boldsymbol{x}^{\boldsymbol{T}}(\boldsymbol{t})\boldsymbol{P}\,\boldsymbol{x}(\boldsymbol{t}))=\boldsymbol{x}^{\boldsymbol{T}}(t)\boldsymbol{Q}\,\boldsymbol{x}(t)+\boldsymbol{u}^{\boldsymbol{T}}(t)\boldsymbol{R}\boldsymbol{u}(t)$ because you could then compute the integral *J* easily using the fundamental theorem of calculus.
4. Assuming 3. above, and that the states tend to zero, because the system after feedback is stable, simplify *J*, and show that minimizing *J* is the same as minimizing ***P*** for a given initial state.
5. Taking the equation in 3. above, differentiate, using the product rule the left side, and substitute in for $\dot{\boldsymbol{x}}(t)$ and $\dot{\boldsymbol{x}}^{\boldsymbol{T}}(t)$ from the state equation.
6. To find the minimum with respect to ***K***, implicitly differentiate the result of 5, and set $\frac{d}{d\boldsymbol{K}}\boldsymbol{P}=\boldsymbol{0}$ and solve for ***K*** as a function of ***P*** and substitute that back into the relation from 5 above, as well as keeping that important result handy.
7. The result of 6 should be the Ricatti equation. The Ricatti equation is actually a quadratic equation for ***P***. ***P*** should be positive definite. Solve it for ***P***.
8. Plug your value for ***P*** into the result for ***K*** in 6 above and you have your state feedback gain.
9. There are two values of ***K***, due to the quadratic nature of the Ricatti equation. Which do you want, or will either work?
    1. I suggest finding the poles. We don't want any with positive real parts.
    2. If you start out with ***Q=0***, what is the ***K***, and what are the poles? What if ***A<0*** or ***A>0***? Does this make sense?
    3. What if ***Q>>R***?

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1 Actually ***Q*** can be non-negative definite, if you push things.
2 For a 1x1 matrix, positive definite means greater than zero.